

Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year		Final Term Exam Date: December 24, 2016 Course: Mathematics 1 – A Duration: 3 hours	
- The Exam consists of one page - Answer All Questions	(تخلفات)	- No. of questions: 4 - Total Mark: 100	
Question 1			
Find y' from the following:		24	
(a) $y = x^3 + 3x^2 + 3x$	(b) $y = (x - \sec x)^{-6}$	(c) $y = \log x - \ln \sin x$	
(d) $y = \tan^3 x + \cos x^3$	(e) $y^3 = x \ln x + \sin y$	(f) $y = t \sin t, x = t \ln t$	
Question 2			
(a) Find the following limits:		8	
(i) $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{1 - x^5}$	(ii) $\lim_{x \rightarrow 0} \frac{\log(1 + x)}{4^x - 3^x}$	(iii) $\lim_{x \rightarrow 0} \frac{x - \tan x}{x^3 + 2x^2}$	(iv) $\lim_{x \rightarrow \infty} \sqrt{\frac{x^8 + 2x}{x + 4x^8}}$
(b) Write the Maclurin's series of the function: $f(x) = \frac{1}{1+2x}$.		4	
(c) Sketch the curve of the function : $f(x) = x^3 - 6x^2$		4	
(d) State and verify Rolle's theorem for, $f(x) = (x - 2)^2$ in interval $[1, 3]$.		4	
(e) Find the integrals : (i) $\int \left(x^3 + \frac{1}{x^3}\right) dx$		(ii) $\int (\sin x - \cos x)^2 dx$	6
Question 3			
(a) Prove that : $4^n - 1$ is divisible by 3 for any integer $n \geq 1$.		6	
(b) Find the sum : $\sum_{r=1}^n (2r - 1)(2r + 1)$		6	
(c) Determine the coefficient of x^{20} in the expansion : $(1 + x + x^2 + x^3 + x^4)(1 - x)^{-4}$		6	
(d) Find the cubic root of the complex number : $z = 1 + i$.		6	
Question 4			
(a) Resolve the fraction : $\frac{x^4 - x^3 - x - 1}{x^3 - x^2}$ into its partial fractions.		6	
(b) Write the first four terms in the expansion : $\sqrt{4 + 2x}$.		6	
(c) Use the Gauss-Jordan algorithm to solve the linear system : $-5x_1 - 2x_2 + 2x_3 = 14, 3x_1 + x_2 - x_3 = -8, 2x_1 + 2x_2 - x_3 = -3$		7	
(d) Show that $x = 1 + 3i$ is a zero to the polynomial $f(x) = x^4 - 2x^3 + 9x^2 + 2x - 10$ and then solve the equation $f(x) = 0$.		7	

Good Luck

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Model Answer

Answer of Question 1

$$(a) y' = 3x^2 + 3^{x^2} \cdot \ln 3 \cdot 2x + 3$$

$$(b) y' = -6(x - \sec x)^{-7} (1 - \sec x \cdot \tan x)$$

$$(c) y' = \frac{1}{\ln 10} \cdot \frac{1}{x} - \frac{\cos x}{\sin x}$$

$$(d) y' = 3 \tan^2 x \cdot \sec^2 x - \sin x^3 \cdot 3x^2$$

$$(e) 3y^2 \cdot y' = 1 + \ln x + \cos y \cdot y'$$

$$(f) y' = \frac{t \cdot \cos t + \sin t}{1 + \ln t}$$

-----24-Marks

Answer of Question 2

$$(a)(i) \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{1 - x^5} = \frac{0}{0} = -\frac{1}{15}$$

$$(ii) \lim_{x \rightarrow 0} \frac{\log(1+x)}{4^x - 3^x} = \frac{0}{0} = \frac{1/\ln 10}{\ln(4/3)}$$

$$(iii) \lim_{x \rightarrow 0} \frac{x - \tan x}{x^3 + 2x^2} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{1 - \sec^2 x}{3x^2 + 4x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{-2 \sec^2 x \cdot \tan x}{6x + 4} = 0$$

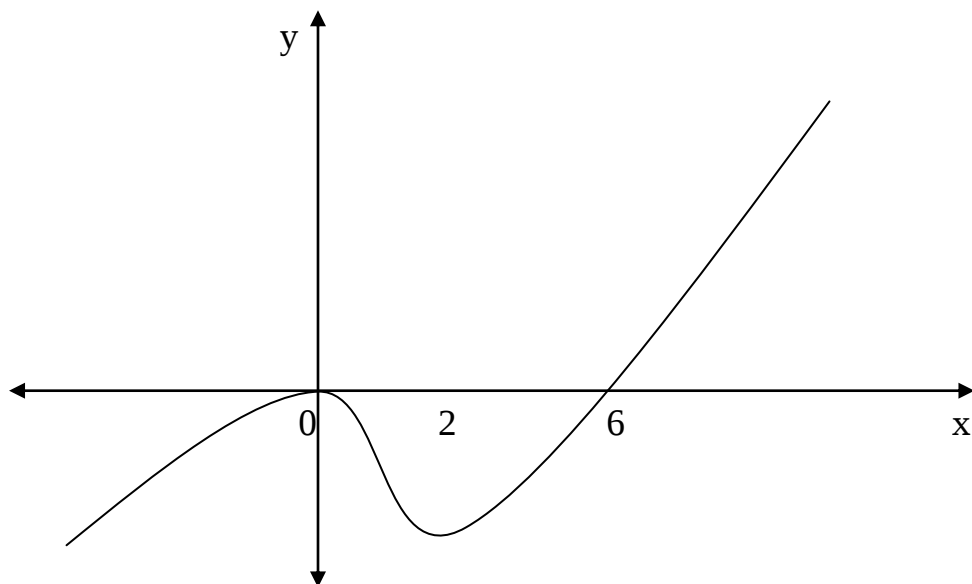
$$(iv) \lim_{x \rightarrow \infty} \sqrt{\frac{x^8 + 2x}{x + 4x^8}} = \frac{\infty}{\infty} = \frac{1}{2}$$

-----8-Marks

$$(b) f(x) = \frac{1}{1+2x} = 1 - 2x + 4x^2 - \dots$$

-----4-Marks

(c)



-----4-Marks

(d) We see that $f(x)$, its derivative $f'(x) = 2(x - 2)$ are continuous in the given interval $[1, 3]$ and $f(1) = f(3) = 1$.

Then $f'(c) = 2(c - 2) = 0$. Hence $c = 2$.

-----4-Marks

$$(e)(i) \int \left(x^3 + \frac{1}{x^3} \right) dx = \frac{1}{4}x^4 - \frac{1}{2}x^{-2} + c$$

$$(ii) \int (\sin x - \cos x)^2 dx = \int (1 - 2 \sin x \cos x) dx = \int (1 - \sin 2x) dx = x + \frac{1}{2} \cos 2x + c$$

-----6-Marks

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Answer of Question 3

(a) Prove that $4^n - 1$ divisible by 3 for any integer $n \geq 1$.

Proof:

STEP 1: For $n=1$ $P(1)$ is true, since $4^1 - 1$ divisible by 3.

STEP 2: Suppose $P(k)$ is true for some $n \geq k \geq 1$, that is $4^k - 1$ divisible by 3

STEP 3: Prove that $P(k+1)$ is true for $n \geq k+1$, that is $4^{k+1} - 1$ divisible by 3

We have $4^{k+1} - 1 = 4 \cdot 4^k - 1 = (3+1)4^k - 1 = (3 \cdot 4^k) + (4^k - 1)$

Which show that $4^{k+1} - 1$ divisible by 3 then $P(n)$ true

(b) Find the sum $\sum_{r=1}^n (2r-1)(2r+1)$

Answer

$$u_r = (2r-1)(2r+1)$$

$$= \frac{(2r-1)(2r+1)(2r+3)}{6} - \frac{(2r-3)(2r-1)(2r+1)}{6} = f(r+1) - f(r)$$

$$S_n = \frac{(2n-1)(2n+1)(2n+3)}{6} + \frac{1}{2}$$

(c) Find coefficient x^{20} in the expansion $(1+x+x^2+x^3+x^4)(1-x)^{-4}$.

Answer

$$(1+x+x^2+x^3+x^4) = \left(\frac{1-x^5}{1-x} \right) (1-x)^{-4} = (1-x^5)(1-x)^{-5}$$

$$= (1-x^5) \left(\sum_{r=0}^{\infty} C_r^{5+r-1} x^r \right) = (1-x^5) \left(\sum_{r=0}^{\infty} C_r^{r+4} x^r \right)$$

Coefficient is $C_{20}^{24} - C_{15}^{19}$

(d) Find the cubic roots for the complex number $z = 1+i$

Solution:

First we put the given complex number $z = 1+i$ in the polar form

$$\therefore 1+i = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \text{ Then}$$

$$(1 + i)^{1/3} = (\sqrt{2})^{1/3} \left[\cos \left(\frac{\pi/4 + 2k\pi}{3} \right) + i \sin \left(\frac{\pi/4 + 2k\pi}{3} \right) \right],$$

$$k = 0, 1, 2$$

at $k = 0$ we determine the first root

$$z_0 = (\sqrt{2})^{1/3} \left[\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right]$$

at $k = 1$ we determine the second root

$$\begin{aligned} z_1 &= (\sqrt{2})^{1/3} \left[\cos \left(\frac{\pi/4 + 2\pi}{3} \right) + i \sin \left(\frac{\pi/4 + 2\pi}{3} \right) \right] \\ &= (\sqrt{2})^{1/3} \left[\cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12} \right] \end{aligned}$$

at $k = 2$ we determine the third root

$$z_2 = (\sqrt{2})^{1/3} \left[\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right]$$

Answer of Question 4

(a) Resolve the fraction $\frac{x^4 - x^3 - x - 1}{x^3 - x^2}$ into its partial fractions.

Solution:

The fraction is an improper fraction. By division

$$\frac{(x^4 - x^3 - x - 1)}{x^3 - x^2} = x - \frac{x+1}{x^3 - x^2} = x - \frac{x+1}{x^2(x-1)} \quad (1)$$

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} \quad (2)$$

$$\frac{x+1}{x^2(x-1)} = \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)}$$

$$\therefore x+1 = Ax(x-1) + B(x-1) + Cx^2 \quad (3)$$

For $x = 0 \Rightarrow 1 = -B \Rightarrow \boxed{B = -1}$

For $x = 1 \Rightarrow 2 = C \Rightarrow \boxed{C = 2}$

For $x = 2 \Rightarrow 3 = 2A + B + 4C \Rightarrow \boxed{A = -2}$

$$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{(x-1)}$$

(b) Use the Gauss-Jordan algorithm to solve each of the following systems of linear equations.

$$-5x_1 - 2x_2 + 2x_3 = 14$$

$$3x_1 + x_2 - x_3 = -8$$

$$2x_1 + 2x_2 - x_3 = -3$$

Solution

The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} -5 & -2 & 2 & 14 \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array} \right)$$

Applying Gauss-Jordan algorithm to this matrix yields

$$\left(\begin{array}{ccc|c} 1 & 2/5 & -2/5 & -14/5 \\ 0 & -1/5 & 1/5 & 2/5 \\ 0 & 6/5 & -1/5 & 13/5 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2/5 & -2/5 & -14/5 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2/5 & 0 & -4/5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array} \right) \text{ the system has unique solution } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$$

(c) Show that the value $x = 1 + 3i$ is a zero to the polynomial

$$f(x) = x^4 - 2x^3 + 9x^2 + 2x - 10 \text{ hence solve the equation } f(x) = 0$$

Solution

Divide the given polynomial by the linear expression $(x - 1 - 3i)$ and we found that the remainder $R = f(x - 1 - 3i) = 0$ this means that the value $1 + 3i$ is a zero of $f(x)$ and $1 + 3i$ is a root for $f(x) = 0$, $1 - 3i$ is another root and

$$f(x) = (x - 1 - 3i)(x - 1 + 3i)Q(x)$$

$$= (x^2 - 2x + 10)Q(x)$$

The polynomial $Q(x)$ is obtained by synthetic division as follows

1	1	-2	9	2	-10
2		2	-10		
-10			0	0	
				-2	10
				0	0

$$\text{Then } f(x) = (x^2 - 2x + 10)(x^2 - 1) = (x^2 - 2x + 10)(x - 1)(x + 1)$$

The roots are $1 \pm 3i, 1, -1$

(d) Write the first four terms in the expansion $\sqrt{4+2x}$

Answer

$$\begin{aligned}\sqrt{4+2x} &= 2\left(1+\frac{x}{2}\right)^{\frac{1}{2}} = 2\left[1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{x}{2}\right)^2+\frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{x}{2}\right)^3+\dots\right] \\ &= 2\left[1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}+\dots\right]\end{aligned}$$

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